Erasure Insertion in Generalized Error Locating Codes*

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Abstract. In this paper we propose a new method of erasures insertion in first outer code of generalized error locating (GEL) code. This method is based on estimations of probability of inner codes syndromes. These probabilities are used in a threshold obtaining algorithm. An optimization of this algorithm is also considered. Numerical results for suggested algorithm for some signal-noise ratios (SNRs) are presented. These results allow us to conclude that first outer codes redundancy can be significantly decreased (especially for low SNRs) using our proposed procedure.

Keywords: GEL codes, threshold, erasures

1 Introduction

Data transmission speeds are significantly growing with transition from 4G to 5G networks. Restrictions on available frequency bandwidth and power stimulate use of high-order modulations and long high-rate codes. Requirements on communication quality demand these codes to achieve error probability of the order of 10^{-15}. That low probabilities make computer simulation of codes' performance implausible and require possibility to assess it analytically.

Since use of soft decision for high-speed information transmission systems can be problematic (i.e. too expensive or requiring too much power), codes having good performance when being hard-decision decoded become more important. In other hand it is widely known that soft-decision decoding results in better coding gain compare with hard-decision one. In this terms codes that allow partial soft and partial hard decoding can be considered as a good candidates for perspective information transmission systems.

Because the energy in the communication channels is severely restricted, the decoding of such signal-code construction should provide a large energy gain (till 11 dB). Thus, this signal-code construction should be based on non-binary code with low redundancy and decoding algorithms with low complexity and the possibility of parallelization. All these requirements are most satisfied by the non-binary low-density parity-check (LDPC) codes and the generalized concatenated code, which special case is the generalized error-locating (GEL) codes. LDPC

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are known to be prone to error floors [1], [2] and it is doubtful whether it is possible to analytically bound their error probability. Reed-Solomon codes have good performance when being both hard- and soft-decision decoded and it is possible to compute error probability (for bounded-distance decoders), but their lengths are strictly restricted.

GEL-code is considered as a possible code-candidate for data transmission systems that require high code rates along with strict requirements on wrong decoding probability, e.g. for the 5G networks. In paper [3] a soft decoding of inner codes of GEL codes was considered. It was shown that such approach can significantly improve a performance of GEL code in terms of maximal achievable rate for fixed input error probability.

In this paper we suggest a method of erasures insertion in first outer code of GEL code. This method is based on estimations of probability of inner codes syndromes. This estimations are calculated using trellis representation of short inner codes. Values obtained that way are employed to estimate an optimal threshold for erasures insertion.

In [7] was proved that in the case of high signal-noise ratios and relatively large number of thresholds soft decision decoding exceeds hard decision one on 3 dB. If the number of thresholds is one and this threshold was chosen optimal, then soft decoding outperforms hard decoding on 1.5 dB. All these results was proved assuming binary-symmertic channel (BSC). Threshold decoding in gaussian channels was considered in paper [8].

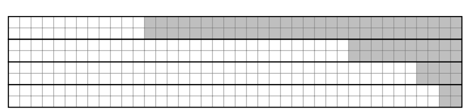
2 Code construction

Let us describe code construction under consideration. Let us refer to it as *normal* GEL-code.

Codeword of *normal* q-ary GEL-code is a matrix $\mathbf{C}$ over $GF(q)$ with size $n_A \times n_B$, where n_A — inner codes length, n_B — outer codes length.

$\mathbf{H}$ is a parity-check matrix of inner codes system $\{\mathcal{C}_j^A\}$. $\mathbf{H}$ is over $GF(q)$, its size is $n_A \times n_A$. Any first r_A rows of this matrix, where r_A is even, comprise $r_A \times n_A$ matrix that is a parity-check matrix of inner code of n_A with $k_A = n_A - r_A$ information symbols. We enumerate these codes in the following way: j-th inner code is a code with $r_A = 2j$. We denote number of information symbols, of parity-check symbols and code distance of these codes as k_j^A, r_j^A and d_j^A respectively. We consider only MDS inner codes. That means that being hard decoded, j-th inner code would be capable of correcting j errors.

Let us denote

$\mathbf{S} = \mathbf{H} \cdot \mathbf{C} =$ 

— a matrix of inner codes' syndromes. Let us group its rows in pairs. We shall denote row pairs *layers* and treat their 2×1 submatrices as symbols over $Q = q^2$.

Then layers are row vectors $\mathbf{s}_j$ of length n_B over $GF(Q)$, $j = \overline{1,l}$. Number of layers $l = n_A/2$ is *order of GEL code*.

Outer codes of *normal* GEL code $\mathcal{C}_j^B$ — codes over $GF(Q)$ of length n_B. They have r_j^B parity-check symbols, rates $R_j^B = 1 - r_j^B/n_B$, $k_j^B = n_B - r_j^B$ information symbols and code distances d_j^B.

GEL-code is a set of matrices $\mathbf{C}$ such that layers $\mathbf{s}_j$, $j = \overline{1,l}$ of matrix $\mathbf{S} = \mathbf{H} \cdot \mathbf{C}$ are codewords of outer codes.

This code is a linear code over $GF(q)$ of length $n = n_A n_B$, it has $r = \sum_j 2r_j^B$ parity symbols. Its rate is $R = \sum_j 2R_j^B/n_A$. Code distance is lower bounded by $d \geq \min_{j=\overline{2,L}}\{d_1^B, d_{j-1}^A d_j^B\}$ [4].

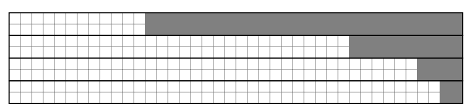
It is worth noting that codeword $\mathbf{C}$ of GEL code doesn't contain codewords of neither outer nor inner component codes. To be more specific, columns of $\mathbf{C}$ contain words of cosets of inner codes. Choice of coset in i-th column is determined by those symbols of outer codes that are in i-th column of $\mathbf{S}$ (in i-th positions of row vectors $\mathbf{s}_j$).

3 Encoding algorithm

The simplest way of encoding GEL code is non-systematic. We will describe it briefly.

Consider information matrix $\mathbf{I}$ over $GF(q)$ that has the same size and partitioned into layers $\mathbf{I}_j$ in the same way as $\mathbf{S}$. First k_j^B symbols of $\mathbf{I}_j$ contain information symbols, last r_j^B are zeros.

$\mathbf{I} =$



In order to obtain a codeword of GEL code we must at first encode layers of $\mathbf{I}$ by outer codes. Outer codes' encoders use information symbols, ignore zeros and return row vectors $\mathbf{s}_j$ that are codewords that correspond to given information symbols. $\mathbf{s}_j$ are components of matrix $\mathbf{S}$.

Let us denote this operation as $\mathbf{S} = Enc_B\{\mathbf{I}\}$.

For obtaining a codeword one must multiply obtained matrix by inverse of $\mathbf{H}$: $\mathbf{C} = \mathbf{H}^{-1} \cdot \mathbf{S}$. As it was said, $\mathbf{H}$ must be non-singular.

$$\mathbf{C} = \mathbf{H}^{-1} \cdot \mathbf{S}$$

The procedure above is the simplest way of describing encoding of GEL code but not the one with minimal complexity. Actually, since we use RS-codes as component codes, it can be performed by existing RS encoders. This also makes possible to dynamically select rates of outer codes and thus tune construction for current channel characteristics.

4 Decoding algorithm

We will briefly describe a decoding procedure considered in [3]. First let us consider a decoding of inner codes only.

4.1 Soft decoding of inner codes

Let us consider transmission of a word of j-th inner code through memoryless channel with soft output. Any symbol c_i of transmitted codeword $\mathbf{c}_j$ would be received as a soft value. This value can be represented as a vector over $\mathbb{R}^q$:

$$\mathbf{v}_i = \big(Pr\{c_i = 0\}, Pr\{c_i = 1\}, \ldots, Pr\{c_i = q-1\}\big)$$

So, the senseword is a vector of soft values. Our further references to these values will omit their internal structure.

Since the transmitted words are members of cosets (including the code itself), decoder of inner code must know the coset it would decode. The decoder goal is to find a word from the coset that is closest to the received word. Thus, the decoder has two inputs: one for coset index and one for received word, and returns a word from the coset. We will denote the decoding operation of j-th inner code as follows:

$$\mathbf{v}_j = Dec_A[\mathbf{s}_j]\{\mathbf{v}\}$$

where $\mathbf{s}_j$ is a coset index, $\mathbf{v}$ is a received vector and $\mathbf{v}_j$ is a decoding result that is a vector over $GF(q)$. Coset index $\mathbf{s}_j$ is essentially a syndrome of transmitted word $\mathbf{c}_j$: $\mathbf{s}_j = \mathbf{H}_j\mathbf{c}_j$.

Generally we don't need to know exact type of decoder to assess performance of GEL code. The only things we are interested in are error probabilities of each code of nested inner codes system for channel under consideration. Let us denote p_{Aj} an error probability of j-th inner code for a channel under consideration. These probabilities are used in [3] for derivation of equations of an upper bound on error probability of the whole GEL construction.

4.2 Estimation of cosets probabilities

Let us consider a first inner code over $GF(q)$ with maximal rate. This code has the following parameters: $(n, k, d) = (n_A, n_A - 2, 3)$. Thus the redundancy of this code is $r_A = 2$ and the number of it's syndromes (cosets) is q^2. Trellis representation of such code is represented in fig. 1.

Trellis decoding of inner codes is considered in [5]. Traditional trellis decoding find the most likely word in each code's coset and the output of this algorithm is the most likely word over all cosets. But our goal is to estimate probabilities of all cosets. This issue can be solved using traditional trellis decoding with some modifications: in traditional case for every state we choose an edge with maximal probability among all q symbol probabilities but for our case we summarize all these probabilities. In this case we calculate probabilities of all words in each coset. This sum equals to probability of that coset.

Thus for $(n_A, n_A - 2, 3)$-code algorithm returns q^2 probabilities $p_{s,1}$, $p_{s,2}$, ..., p_{s,q^2} of all q^2 cosets.

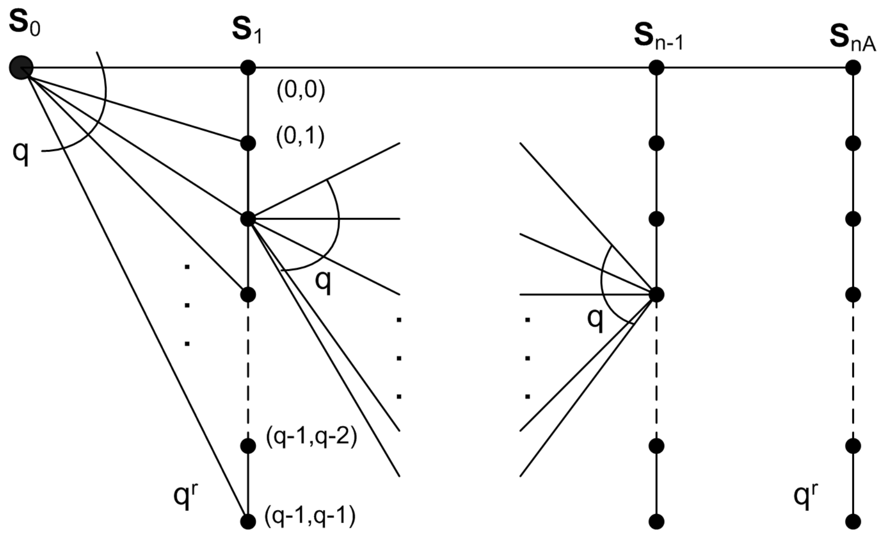


Fig. 1. Trellis structure.

4.3 Erasures insertion

Let us consider $(n_A, n_A - 2, 3)$ inner code and transmission of it's codewords $\mathbf{c}$ through memoryless channel with soft output discussed above. Let $\mathbf{v}$ is a received soft word. Let $\mathbf{v}_h$ is a hard decision of $\mathbf{v}$. For this word we collect the following statistic:

$$\begin{pmatrix} Idx(\mathbf{v}_h \mathbf{H}_1^T = \mathbf{c}\mathbf{H}_1^T) \\ p_{s,max} \\ p_{s,max2} \end{pmatrix}$$

where $Idx(\mathbf{v}_h \mathbf{H}_1^T = \mathbf{c}\mathbf{H}_1^T) = 1$ if $\mathbf{v}_h \mathbf{H}_1^T = \mathbf{c}\mathbf{H}_1^T$ and $Idx(\cdot) = 0$ otherwise; $p_{s,max}$ is a probability of most likely coset and $p_{s,max2}$ is a probability for second likely one.

Let us choose relatively large $N \in \mathbf{N}$ and collect a statistics discussed above for $\mathbf{c}_1$, ..., $\mathbf{c}_N$. We obtain the following $3 \times N$ matrix:

$$\mathbf{M} = \begin{pmatrix} Idx(\mathbf{v}_{h1} \mathbf{H}_1^T = \mathbf{c}_1 \mathbf{H}_1^T) & \dots & Idx(\mathbf{v}_{hN} \mathbf{H}_1^T = \mathbf{c}_N \mathbf{H}_1^T) \\ p_{s,max}^{(1)} & \dots & p_{s,max}^{(N)} \\ p_{s,max2}^{(1)} & \dots & p_{s,max2}^{(N)} \end{pmatrix}$$

Let us choose threshold $L \in [0; 1)$:

$$if \;\; p_{s,max}^{(i)} - p_{s,max2}^{(i)} < L \;\; then \;\; \mathbf{v}_{hi} \mapsto \star.$$

The latest expression means that if probabilities of two most likely cosets are relatively close to each other then i-th symbol of first outer code is considered as a erasure. For this setup we can calculate two probabilities for first outer code: error $p_t(L)$ and erasure $p_\tau(L)$ ones. They certainly depend on the threshold L:

$$p_t(L) = \frac{|i : \{Idx(\cdot) = 0\} \cap \overline{\{\mathbf{v}_{hi} \mapsto \star\}}|}{N}$$

$$p_\tau(L) = \frac{|i : \{\mathbf{v}_{hi} \mapsto \star\}|}{N}$$

It is evident that

$$p_t(0) \geq p_t(L) \, \forall \; L > 0$$

and

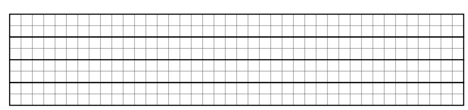
$$L_1 > L_2 \Rightarrow p_\tau(L_1) \geq p_\tau(L_2).$$

Probability $p_t(L)$ is a number of inner codewords which were received with errors and were not erased divided by N, $p_t(L)$ is a number of erased codewords divided by N. If $N = n_B$ then set $E(L) = \{i : \mathbf{v}_{hi} \mapsto \star\}$ is a number of positions in first outer code which should be erased. Similarly, this set depends on threshold L. Thus, the main objective is to compute optimal threshold L.

4.4 Soft decoding GEL code with erasures insertion

First let us describe a decoding of whole GEL code construction. Let us consider the transmission of constructed above code word $\mathbf{C}$ of GEL-code of order l over memoryless channel with soft output.

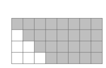
Then the received word is a matrix of size $n_A \times n_B$ with soft values:

$\mathbf{V} =$ 

Decoding algorithm is iterative, number of iterations equal code order l. Each iteration has two main steps: decoding of outer code and decoding of inner codes. We consider decoding during j-th iteration under assumption that decoding during all previous $j-1$ iterations was errorless.

Before decoding we perform the following procedure: according with $E(L)$ we mark the corresponding symbols of first outer code as a erasures. Thus this code corrects not only errors but both errors and erasures. Other outer codes are assumed to correct errors only.

Let us denote $\mathbf{H}_j$ parity-check matrix of j-th inner code, it contains first j layers ($2j$ rows) of $\mathbf{H}$:

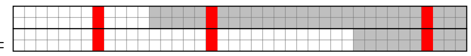
$\mathbf{H}_j =$ 

Then decoding algorithm for j-th iteration ($j = \overline{1, l}$) is as follows:

- Prior to decoding $\mathbf{V}_0$ is a hard decision of receiver word $\mathbf{V}$. (It can be thought as a result of decoding 0-th inner code.)

- The word $\mathbf{V}_{j-1}$ obtained during previous iteration is left multiplied by $\mathbf{H}_j$. The result is a matrix of inner codes' syndromes for j-th inner code(s):

$$\mathbf{S}'_j = \mathbf{H}_j \cdot \mathbf{V}_{j-1} =$$



  It will contain some columns that differ from transmitted ones. It means that they contain errors. An example set of such columns is marked by red.
- Then decoder of $\mathcal{C}^B_j$ corrects errors (and erasures if $j = 1$) in $\mathbf{S}'_j$:

$$\mathbf{S}''_j = Dec_B\{\mathbf{S}'_j\}$$

  The result is that under condition that outer codes have sufficient error correction capability to correct all errors (and erasures if $j = 1$) in $\mathbf{S}'_j$, columns of $\mathbf{S}''_j$ would give correct cosets for decoding inner codes.
  If outer code returns decoding refusal then decoding algorithm stops and refusal is returned.
- Obtained matrix $\mathbf{S}''_j$ is used by j-th inner codes to correct errors in $\mathbf{V}$

$$\mathbf{V}_j = Dec_A[\mathbf{S}''_j]\{\mathbf{V}\}$$

- After correction of errors by last, l-th layer, decoding ends. Matrix $\mathbf{V}_l$ is the result of decoding.

5 Estimation of optimal threshold

Let us consider first outer MDS code with the following parameters: $(n_B, k^B_1, d^B_1 = n_B - k^B_1 + 1 = r^B_1 + 1)$. For this code we have probability of error $p_t(L)$ and probability of erasure $p_\tau(L)$, then probability of wrong decoding can be calculated as follows:

$$p_f = \sum_{i=0}^{n_B} \sum_{j=r^B_1+1-2i}^{n_B-i} \binom{n_B}{i,j} p_t(L)^i p_\tau(L)^j (1 - p_t(L) - p_\tau(L))^{n_B-i-j} \qquad (1)$$

Assume that output probability p_{targ} is fixed. Then for given p_{targ}, $p_\tau(L)$, $p_t(L)$, n_B we can calculate a minimal redundancy r^B_{min} that is required to satisfy the condition: $p_f \leq p_{targ}$ [6]. The algorithm for finding an optimal L is following:

Data: n_B, $\mathbf{M}$, p_{targ}, δ
Result: $L_{optimal}$, r^B_{min}
$r^B_{min} \leftarrow n_B$;
for $L = 0 : \delta : 1$ **do**
  Calculate $p_\tau(L)$, $p_t(L)$;
  Find $r^B_{min}(L)$ according with (1);
  if $r^B_{min} < r^B_{min}(L)$ **then**
    $r^B_{min} \leftarrow r^B_{min}(L)$;
    $L_{optimal} \leftarrow L$;
  end
end

where δ is an accuracy of threshold calculation. This algorithm returns an optimal threshold $L_{optimal}$ which provides the minimal possible redundancy r^B_{min} of first outer code. According with this threshold one can construct the set $E(L) = \{i : \mathbf{v}_{hi} \mapsto \star\}$ of erased positions in first outer code.

6 Numerical results

In this section we use the described above method of erasures insertion and threshold optimization to construct first outer code for the given probability of the wrong decoding $p_{targ} = 10^{-15}$ and energy per symbol E_s/N_o for the AWGN channel with quadrature amplitude modulation (QAM) of order $M = 16$. We consider the performance (in terms of maximal achievable code rate for given input and output error probabilities) of our proposed constructions.

In this paper we implement a soft maximum likelihood decoding (trellis decoding) for inner code with parameters $n_A = 6, k_A = 4$. Inner codes are shortened RS-codes over $GF(2^4)$. The length of outer codes is $n_B = 256$. Outer codes are RS-codes over $GF(2^8)$. Thus the lengths of obtained GEL code is 6144 bits and the number of layers is $l = 3$.

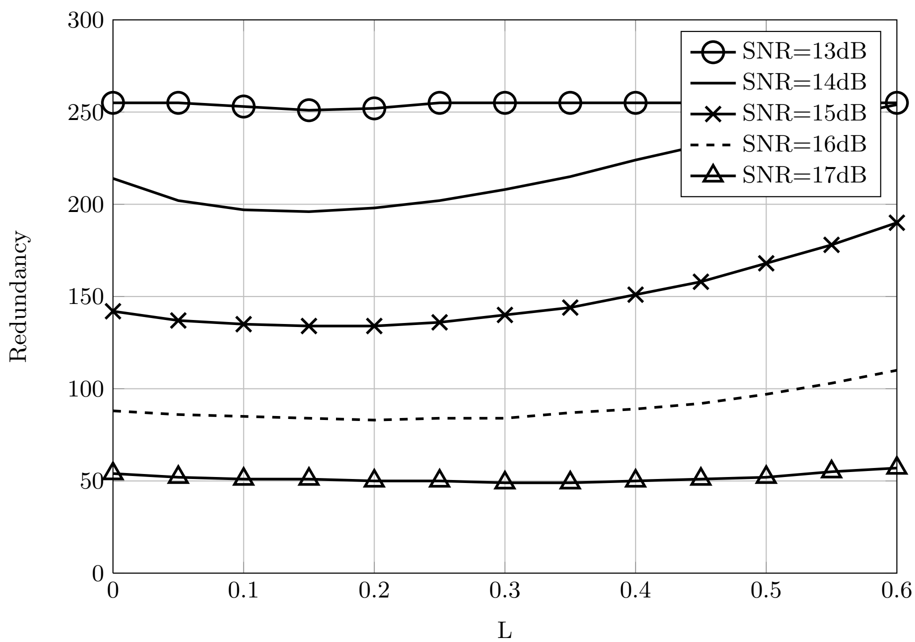


Fig. 2. Redundancy versus threshold, $n_B = 255$, $p_f = 10^{-15}$

Dependence between threshold L and redundancy of first outer code is presented in fig. 2. One can notice that every curve has global minimum r^B_{min}. Value

L that corresponds to this minimum is $L_{optimal}$. The following table presents dependence between $r^B_{min}(L_{optimal})$, $r^B_{min}(0)$ (redundancy of code that corrects errors only), $L_{optimal}$ and SNR.

SNR	$L_{optimal}$	$r^B_{min}(0)$	$r^B_{min}(L_{optimal})$	difference
13	0.15	255	251	4
14	0.15	214	196	18
15	0.15	142	134	8
16	0.2	88	83	5
17	0.3	54	49	5

It can be noticed that for every SNR there is an $L_{optimal}$ that effects in decreasing of redundancy of first outer code. For $SNR = 14dB$ this reduction is maximal. Graphical representation of table 6 is presented in fig. 3 and 4.

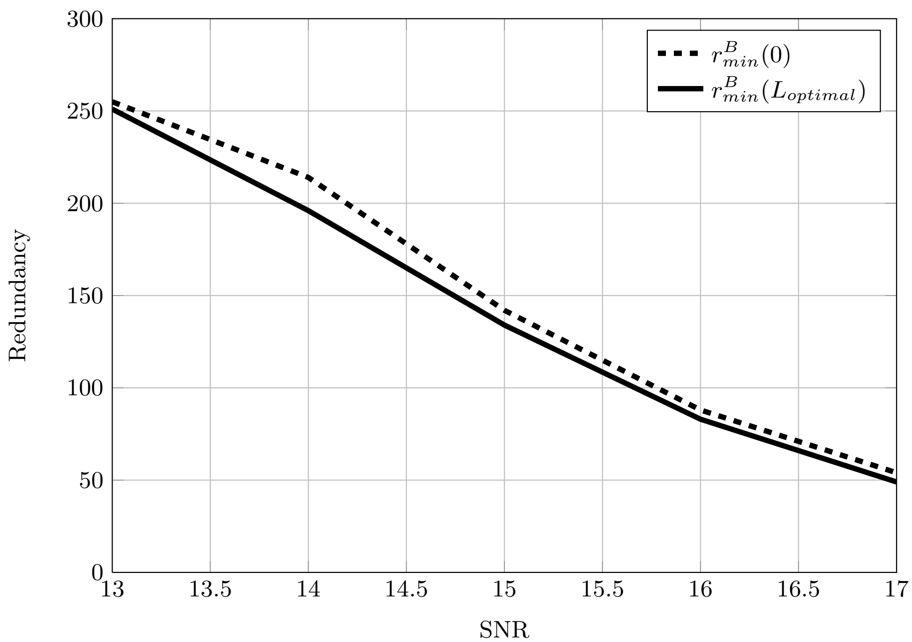


Fig. 3. $r^B_{min}(0)$ versus $r^B_{min}(L_{optimal})$, $n_B = 255$, $p_f = 10^{-15}$

7 Conclusion

In this paper we proposed a new method of erasure insertion in first outer code of GEL code where inner code are short and decoded using soft maximum-

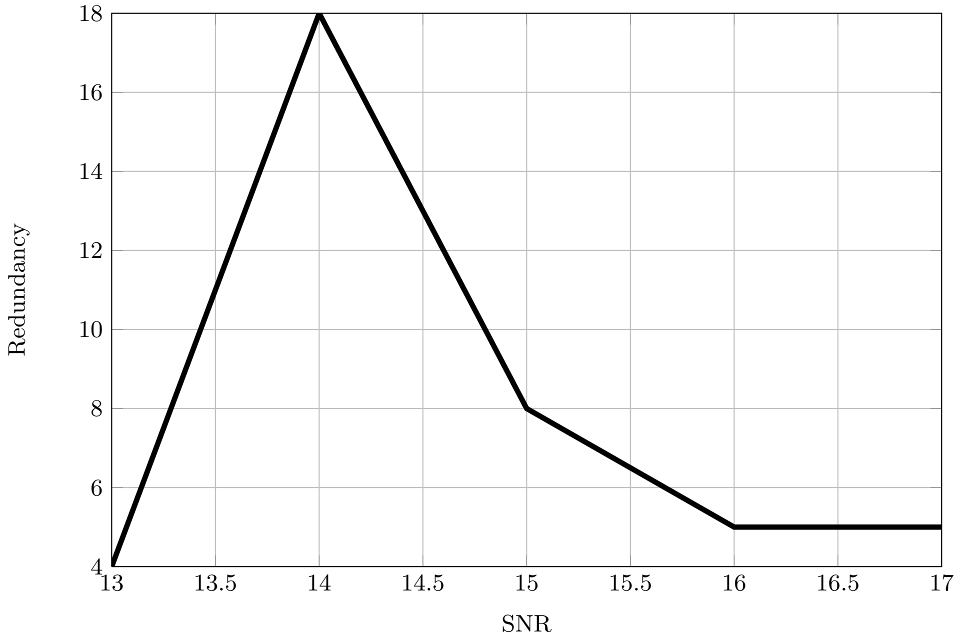


Fig. 4. Dependence between $r^B_{min}(0) - r^B_{min}(L_{optimal})$ and SNR, $n_B = 255$, $p_f = 10^{-15}$

likelihood decoder and outer codes are decoded using conventional bounded minimum-distance decoder. The method of erasure insertion is based on estimations of probability of inner codes syndromes.

Numerical results allow us to conclude that our proposed method allow to decrease a redundancies of first outer codes thus increasing a code rate of GEL code construction.

References

1. Richardson, T.: Error Floors of LDPC Codes. Proc. of the annual Allerton conf. on communication control and computing. 1998. 41, 3, 1426–1435.
2. Vontobel, P. O., Koetter, R.: Graph-Cover Decoding and Finite-Length Analysis of Message-Passing Iterative Decoding of LDPC Codes. arXiv:cs/0512078v1 [cs.IT].
3. Zhilin, I., Ivanov, F., Zyablov V.: Generalized Error Locating Codes with Soft Decoding of Inner Codes. In proc. of 21th European Wireless (EW) conference, 2015 (to appear).
4. Blokh, E., Zyablov, V.: Generalized concatenated codes. Moscow, U.S.S.R: Svyaz', 1976. In Russian.
5. Wolf, J.K.: Efficient maximum likelihood decoding of linear block codes using a trellis. Information Theory, IEEE Transactions on. 1978. 24, 1, 76–80.
6. Zhilin, I., Kreshchuk, A., Zyablov, V.: Generalized Error Location Codes and Minimization of Redundancy for Given Input and Output Error Rates. Journal of Communications Technology and Electronics (to appear).

7. Fourney, G.: Generalized minimum distance decoding. Information Theory, IEEE Transactions on. 1966. 12, 125–131.
8. Senger, C.: Generalized minimum distance decoding with arbitraty error/erasure tradeoff. PhD thesis. Ulm University, Germany. 2011.