Development of numerical procedure for control of connected Markov chains *

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Abstract. The system of controlled time-inhomogeneous Markov chains (MC) is considered. The principal problem is the "curse of dimension" which appears here as the necessity of solving the system of ordinary differential equations of high dimension. Moreover, even the development of the system itself is a serious issue since these equations are linked and the standard parallelization approach developed in existing software packages are not very effective. Meanwhile, one can observe that the minimization procedure needed for the right hand side of this system may be easily parallelized since for each equation the minimization procedure may be realized independently. As an example we consider the management of linked dams under seasonal random inflows/outflows and customers' demands. The current state of each dam is the state of continuous-time Markov chain corresponding to the water level. So the state of the dams system is represented in tensor form. The connection of Markov chains is due to the controlled flow between dams. The aim of the control is to keep balance under the natural perturbation and at the same time to satisfy the customers' demands. The general approach to the solution is based on the dynamic programming method which leads to the solution of Bellman type equation in tensor form. This equation may be reduced to the system of ordinary differential equations. Here we suggested the automatic procedure of this system generation and an approach to the minimization which may be realized for each state independently.

Keywords: connected Markov chain, stochastic control, tensor, dynamic programming

1 Introduction

The natural resource management and control in various areas such as forestry, fisheries, pest control, water flow in hydrological systems and so on, may be

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treated as Markov decision processes (MDP). There exists extensive literature on this topic [1], [2]. The controlled continuous time Markov chain (MC) model approximates the continuous state space by discretized one where the control is responsible for the transition intensity from one state to another. The advantage of the continuous-time MDP with respect to the diffusion models is in the relative simplicity for the solution of control problems. That is because the dynamic programming equation for MDP may be reduced to the system of ordinary differential equation instead of nonlinear partial differential equation of high dimension in the case of diffusion model. However, the good approximation needs the large number of states and thereby the large number of differential equations which are weakly linked and do not admit simple parallelization. An attempts to solve more or less serious problems with the aid of standard mathematical packages shows [3] that the existing parallelization tools could improve the calculation speed in 3-4 times and is not applicable to the number of states more than one hundred. Even the derivation of the system of these ordinary equations still a big issue since each equation contains the minimization operation in the right-hand side (RHS). The aim of this work is the development of the approach to the automatic derivation of the RHS on the basis of martingal representation of the MC. Here the connected MCs are considered so instead of vector representation the tensor one is needed. The dynamic programming equation is presented in tensor form and an effective numerical procedure for derivation of the equations in the equivalent system of ordinary differential equations had been proposed. For solution of this system one can use the next important observation so that for minimization of its RHS one can minimize each equation separately by using the minimization tool for problem of rather small dimension. It discovers a way to the genuine parallelization of the numerical solution.

The model of three dams system is considered as an example, where controls are responsible for pour from one dam into another if necessary either to prevent overflow or to satisfy the customers demands in better way. Such models had been considered earlier [3], [4], [5] and demonstrate the possibility to solve rather complicated problems where it is necessary to keep balance and to satisfy the customers' demands with the aid of the water price controls and linkage between dams. The analogous models are applicable to the data transmission problems where buffers and incoming/outcoming packages may be considered in the same framework of flows [6], [7], [8], [9] and the problem of keeping balance is formulated as congestion avoidance.

The aim of this article is to develop the optimal control for MC of high dimension. The principal computational difficulty is that the number of states cannot be reduced and in reasonable case is about few thousands. The control has to be defined for each state and time. Here an approach to the software which admits the parallelization has been suggested, it presumes:

- develop the automatic generation of the system of dynamic programming equations;
- realize the calculation of the minimum in RHS of the above system in parallel way;

- use the numerical solution of the system by Euler method with reasonable time increment since the high accuracy is not necessary for such approximation;
- and finally to develop the data structure for the solution and controls.

2 Model of controllable connected Markov chain

Each of controllable Markov chain (numbered as i^{th}) has N^i possible states and is described by the following stochastic differential equation [10]

$$X_t^i = X_0^i + \int_0^t A_i(s, u(s)) X_s^i ds + W_t^i, \tag{1}$$

where $X_t^i \in \mathcal{S}^i = \{e_0, ..., e_{N^i}\}$, X_0^i is the initial state of the i^{th} Markov chain. The $(N^i + 1 \times N^i + 1)$ matrix $A^i(t, u)$ is the generator of i^{th} Markov chain, matrix valued function A^i is assumed to be continuous on $(t, u) \in [0, T] \times U$, where $T < \infty$ and U is a compact set in $\mathbb{R}^m$. The process W_t^i is a square integrable martingale with bounded quadratic variation [10].

The general theory of controlled connected Markov chain is presented in [6], [7], where the joint state of Markov chain $\mathbf{X} = \{X^1|X^2|...|X^d\}$ be described as tensor product of vectors $\mathbf{X} = X^1 \otimes X^2 \otimes ... \otimes X^d$, where $X^i \in \mathcal{S}^i$. All processes are defined on the probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$.

Assumption 1. The $(N^i + 1 \times N^i + 1)$ matrices $A^i(t, u)$ with entries $a^i_{k,l}(t, u)$ constitute a family of time-dependent generators, that is,

1. $a^i_{k,l}(t, u) \geq 0$, for $k \neq l$;
2. $a^i_{l,l}(t, u) = -\sum\limits_{k \neq l} a^i_{k,l}(t, u)$;
3. the control parameter $u \in U$, where U is some compact set in complete metric space and $A^i(t, u)$ is continuous on $[0, T] \times U$.
4. For given $u \in U$ and the initial distribution $\mathbf{P}(0)$ of $\mathbf{X}(0)$ the probability column tensor $\mathbf{P}(t) = (P^1(t)|...|P^d(t))$, satisfies the Kolmogorov forward equation

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{A}(t, u) \otimes \mathbf{P}(t). \tag{2}$$

where $P^i(t)$ is a vector of probability distribution of i^{th} Markov chain. Here the sign $\otimes$ means the tensor product of the tensor $\mathbf{A}$ consisting of matrices A^i and $\mathbf{P}$ is the tensor consisting of the distributions of particular Markov chains.

We also make the following standard assumptions about the controls $u(t)$.

Assumption 2. The set of admissible controls, $u(\cdot)$ is the set of $\mathcal{F}_t^{\mathbf{X}}$-predictable controls taking values in U.

Remark 1. Assumption 2 ensures that if the number of jumps of the joint connected Markov chain up to the current time $t \in [0,T]$ is $\mathcal{N}_t$, τ_k is the time of the k^{th} jump and

$$(\mathbf{X}^i)_0^t = \{(\mathbf{X}_0^i, 0), (\mathbf{X}_1^i, \tau_1), ..., (\mathbf{X}_{\mathcal{N}_t}^i, \tau_{\mathcal{N}_t})\}$$

is the set of states and jump times, then for $\tau_{\mathcal{N}_t} \leq t < \tau_{\mathcal{N}_{t+1}}$ the controls $u(t) = u(t, (\mathbf{X})_0^t)$ are measurable with respect to t and $(\mathbf{X})_0^t$, where $\mathbf{X}_0^t = (X^1)_0^t \otimes (X^2)_0^t \otimes \ldots \otimes (X^d)_0^t$ [6].

2.1 Performance criterion

Let $f_0(s, p(s), \mathbf{X}_s)$ be the running cost function when the connected Markov chain is in state $\mathbf{X}_s$ at time $s \in [0,T]$. Then a general performance criterion to be minimized has the tensor form

$$J[u(\cdot), \mathbf{X}(\cdot)] = \mathbb{E}\left[\phi_0(\mathbf{X}_T) + \int_0^T f_0(s, p(s), \mathbf{X}_s) ds\right]. \tag{3}$$

Here $\phi_0(\mathbf{X}_T) = \phi_0 \otimes \mathbf{X}_T$ and $f_0(s, p(s), \mathbf{X}_s) = \mathbf{f}_0(s, u(s)) \otimes \mathbf{X}_s$ with ϕ_0 and $\mathbf{f}_0(s, u(s))$ are the tensors of the order d.

Assumption 3. For each $\mathbf{X} \in \mathcal{S} = \mathcal{S}^1 \otimes \mathcal{S}^2 \otimes ... \otimes \mathcal{S}^d$, the elements of $f_0(s,u))$ are bounded below and continuous on $[0,T] \times U$.

2.2 Tensor form of the value function

The value function of connected Markov chain is a function which gives minimum total cost for connected Markov chain starting at time $t \in [0,T]$ and state $\mathbf{X}_t = \mathbf{X} \in \mathcal{S}$. It has the form

$$V(t, \mathbf{X}) = \inf_{u(\cdot)} J[u(\cdot), \mathbf{X}(\cdot) | \mathbf{X}_t = \mathbf{X}],$$

where

$$J[u(\cdot), \mathbf{X}(\cdot) | \mathbf{X}_t = \mathbf{X}] = \mathbb{E}\left[\phi_0(\mathbf{X}_T) + \int_t^T f_0(s, u(s), \mathbf{X}_s) ds \middle| \mathbf{X}_t = \mathbf{X}\right]. \tag{4}$$

We now represent $V(t, \mathbf{X})$ as $V(t, \mathbf{X}) = \phi(t) \otimes \mathbf{X}$, where $\phi(t)$ is a tensor of the order d with measurable components.

2.3 Dynamic programming equation in tensor form

Further we generalize the approach (see [8], Thm. 2.8) to controlled MC in the tensor form. Let $\hat{\phi}(\mathbf{t})$ be of the same form as $\phi(t)$, and define the *dynamic* *programming equation* with respect to $\hat{\phi}(\mathbf{t})$

$$\frac{d\hat{\phi}(t)\otimes\mathbf{X}}{dt} = -\min_{u\in U}\Big\{\hat{\phi}(t)\otimes\Big[A^1(t,u)X^1\otimes X^2\otimes...\otimes X^d+ \\ X^1\otimes A^2(t,u)X^2\otimes...\otimes X^d+\ldots+ \\ X^1\otimes X^2\otimes...\otimes A^d(t,u)X^d\Big]+\mathbf{f_0}(t,u)\otimes\mathbf{X}\Big\} = \\ -\min_{u\in U} H(t,\hat{\phi}(t),u,\mathbf{X}) = -\mathcal{H}(t,\hat{\phi},\mathbf{X}) \tag{5}$$

with boundary condition $\hat{\phi}(T)=\phi_0$ [8], [10], [11]. Since $H(t,\hat{\phi},u,\mathbf{X})$ is continuous in (t,u) and affine in $\hat{\phi}$, for any $(t,\mathbf{X})\in[0,T]\times\mathcal{S}$, $\mathcal{H}(t,\hat{\phi},\mathbf{X})$ is Lipschitz in $\hat{\phi}$.

Proposition 1. *With Assumption 3 held equation (5) has a unique solution on* $[0,T]$.

Remark 2. If we now let $\mathbf{X}=\bigotimes_{k=1}^{d} e_{(i_k)}$, $k=1,...,d$, then we get a system of ODE's

$$\frac{d\hat{\phi}_{\mathbf{i_1},\mathbf{i_2},..,\mathbf{i_d}}(\mathbf{t})}{dt} = -\mathcal{H}\left(t,\hat{\phi}(t),\bigotimes_{k=1}^{d} e_{(i_k)}\right), \\ i_1=1,...,N^1, i_2=1,...,N^2,...,i_d=1,...,N^d. \tag{6}$$

The simple generalization of the Thm. 2.8 [8] gives the following characterization of the optimal control.

Theorem 1. *Let $\hat{\phi}(t)$ be the solution of the system of equations (6), then for each $(t,\mathbf{X})\in[0,T]\times\mathcal{S}$ there exists $u_0(t,\mathbf{X})\in U$ such that $H(t,\hat{\phi}(t),u,\mathbf{X})$ achieves a minimum at $u_0(t,\mathbf{X})$. Then*

1. *There exists an $\mathcal{F}_t^{\mathbf{X}}$-predictable optimal control, $\hat{u}(t,\mathbf{X}_0^t)$ such that $V(t,\mathbf{X}) = J[\hat{u}(\cdot)\big|\mathbf{X}_t=\mathbf{X}] = \hat{\phi}(t)\otimes\mathbf{X}$.*
2. *The optimal control can be chosen as Markovian, that is*

$$\hat{u}(t,\mathbf{X}_0^t) = u_0(t,\mathbf{X}_{t-}) = \underset{u\in U}{argmin}\, H(t,\hat{\phi}(t),u,\mathbf{X}_{t-}).$$

3 Three dams system model

In this article we consider the case of three dams.

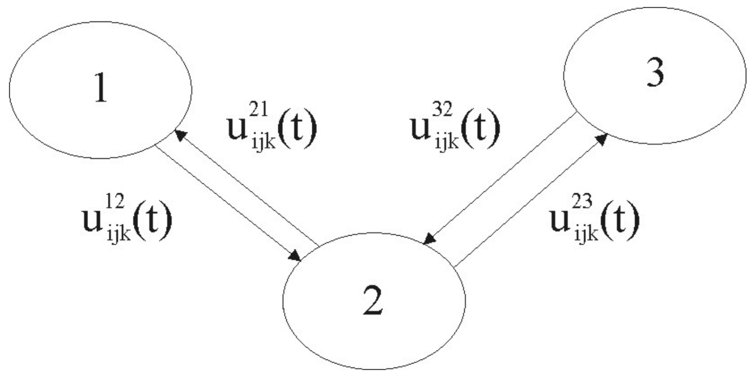

Fig. 1. The topology of three dams system model in the example

3.1 The graph of the dams system

The graph of the dams system is shown on Fig. 1, some of them are linked and accept the water transportation from one to another as shown by arrows. Each pipeline accepts the transportation from m^{th} dam to n^{th} which provides the intensity of transmission $u^{mn}_{ijk}(t)$ depending on current time t and the current state of dams system $X^1 = i, X^2 = j, X^3 = k$. The intensity of transmission satisfy the constraints

$$0 \leq u^{mn}_{ijk}(t) \leq u^{mn}_{max}, \forall\{i, j, k\}. \tag{7}$$

This form of constraints admits the water transportation in both side of pipeline.

Let us represent the level of a dam by discretizing the volume of the i^{th} dam into N^i levels [12] and denote the level at any time $t \in [0, T], T < \infty$, by an integer valued random variable $L^i_t \in \{0, ..., N^i\}$. Each level is represented by the set of unit vectors $S^i = \{e_0, e_1, \ldots, e_{N^i}\}$ in R^{N+1}, then one can define a random vector $X_t \in S$ on $[0, T]$ corresponding to this level at $t \in [0, T]$. It means that $\mathbb{1}\{L_t = i\} = \mathbb{1}\{X_t = e_i\}$ where $\mathbb{1}\{\cdot\}$ is an indicator function.

The use of controlled MC models for dams management had been considered in [4] and for congestion avoidance control [9].

3.2 Inflows and outflows

Assume that the process of inflows to the dam can be approximated by a time-inhomogeneous Poisson process, I_t. The natural inflow for i^{th} dam is the result of rain events which occur with intensity $\lambda^i(t)$. The current inflow to the i^{th} dam is the result of the rain inflow and possible transportation from other dams.

The semi-martingale representations of $I_t^i, i = 1, 2, 3$ are:

$$
\begin{aligned}
I_t^1 &= \int_0^t (\lambda^1(s) + u^{21}(s))\mathbb{1}\{L^1(s) < N^1\}ds + M_t^1 \\
I_t^2 &= \int_0^t (\lambda^2(s) + u^{12}(s) + u^{32}(s))\mathbb{1}\{L^2(s) < N^2\}ds + M_t^2 \\
I_t^3 &= \int_0^t (\lambda^3(s) + u^{23}(s))\mathbb{1}\{L^3(s) < N^3\}ds + M_t^3
\end{aligned}
\tag{8}
$$

where $M_t^i, i = 1, 2, 3$ are a square-integrable martingales [10].

Outflow from each dam consists of natural losses due to evaporation, the consumption of the dam users and possible transportation to other dams. The natural losses of each dam are described by a general counting process with intensity $\mu^i(t), i = 1, 2, 3$. Customers' demands for i^{th} dam are described by a general counting process with intensity $w^i(t), i = 1, 2, 3$.

The outflow from i^{th} dam is the counting process O_t^i which depends on the current state of the dam. The semi-martingale representations of O_t^i are:

$$
\begin{aligned}
O_t^1 &= \int_0^t (\mu^1(s) + w^1(s) + u^{12}(s))\mathbb{1}\{L^1(s) > 0\}ds + M_t^4 \\
O_t^2 &= \int_0^t (\mu^2(s) + w^2(s) + u^{21}(s) + u^{23}(s))\mathbb{1}\{L^2(s) > 0\}ds + M_t^5 \\
O_t^3 &= \int_0^t (\mu^3(s) + w^3(s) + u^{32}(s))\mathbb{1}\{L^3(s) < N > 0\}ds + M_t^6
\end{aligned}
\tag{9}
$$

where $M_t^i, i = 4, 5, 6$ are a square-integrable martingales.

The controls satisfy the constraints

$$
\begin{aligned}
&0 \le u^{12}_{i_1,i_2,i_3}(t) \le 4, \quad 0 \le u^{21}_{i_1,i_2,i_3}(t) \le 4, \\
&0 \le u^{23}_{i_1,i_2,i_3}(t) \le 2, \quad 0 \le u^{32}_{i_1,i_2,i_3}(t) \le 4 \quad \forall t, \forall\{i_1, i_2, i_3\}.
\end{aligned}
\tag{10}
$$

The intesities of inflows, natural outflows as well as the customers demands are choosen as follows:

$$
\begin{aligned}
&\lambda^1(t) = -\cos(2\pi t) + 10, \quad \mu^1(t) = \sin(2\pi t + 5/12\pi) + 3, \\
&\lambda^2(t) = -2\cos(2\pi t) + 14, \; \mu^2(t) = 2\sin(2\pi t + 5/12\pi) + 6, \\
&\lambda^3(t) = -0.5\cos(2\pi t) + 6, \; \mu^3(t) = 0.5\sin(2\pi t + 5/12\pi) + 2.
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
w^1(t) &= \sin(2\pi t + 1/4\pi) + 7, \\
w^2(t) &= 2\sin(2\pi t + 1/4\pi) + 8, \\
w^3(t) &= 0.5\sin(2\pi t + 1/4\pi) + 4.
\end{aligned}
\tag{12}
$$

Remark 3. As usual, we assume that I_t and O_t are processes whose jumps do not occur at the same time. This implies that the mutual quadratic variation, $\langle M^i, M^j\rangle_t = 0, i \neq j$.

Specifically, we define $\mathbf{X} = X^1 \otimes X^2 \otimes X^3$, where $X_t^i \in \mathcal{S}^i, i = 1, \ldots, 3, t \in [0,T], T < \infty$, as a controlled jump Markov process with piecewise constant right-continuous paths.

The corresponding generators of the MCs, that are $A^i(t,u)$, have been defined on the basis of processes $I_t^i, O_t^i, i = 1,2,3$ [8], [9]. For each state of the system $\{i_1, i_2, i_3\}$ and time $t \in [0,T]$ it is necessary to find

$$\left\{u^{12}_{i_1,i_2,i_3}(t), u^{21}_{i_1,i_2,i_3}(t), u^{23}_{i_1,i_2,i_3}(t), u^{32}_{i_1,i_2,i_3}(t)\right\}. \tag{13}$$

3.3 Criterion of optimization

The aim is to find out the controls (13) which provides the best balance between the natural inflows/outflows and the customers' demands. The criteria have the form of quadratic integral functionals which take into account the current squared difference between the inflow and the outflow for each dam

$$\begin{aligned}
f_0^1(t,u) &= \Big(\lambda^1(t) - \mu^1(t) - w^1(t) - u^{12}_{i_1,i_2,i_3}(t) + u^{21}_{i_1,i_2,i_3}(t)\Big)^2 \\
f_0^2(t,u) &= \Big(\lambda^2(t) - \mu^2(t) - w^2(t) + u^{12}_{i_1,i_2,i_3}(t) - u^{21}_{i_1,i_2,i_3}(t) \\
&\quad - u^{23}_{i_1,i_2,i_3}(t) + u^{32}_{i_1,i_2,i_3}(t))\Big)^2 \\
f_0^3(t,u) &= \Big(\lambda^3(t) - \mu^3(t) - w^3(t) + u^{23}_{i_1,i_2,i_3}(t) - u^{32}_{i_1,i_2,i_3}(t)\Big)^2
\end{aligned} \tag{14}$$

3.4 Numeric procedure

To determine the control variables (13) one needs to solve the system of ordinary differntial equations (5). Since the dimension of this system is large enough to be implemented with the standard numerical procedures one can suggest the following sequence of steps for solution of this system by Euler method, which provides the possibility to balance between the number of calculations and the accuracy:

1. define the equations of the RHS of (5) and substitute the functions $\lambda^i(t), \mu^i(t), w^i(t)$ from (11), (12);
2. solve system (5) in backward time by Euler method. For that use $\phi_{i_1,i_2,i_3}(t)$ found out on the preceding step of the integration (for initial step use the terminal values $\phi_{i_1,i_2,i_3}(T)$. Thereby, define the RHS of (5) and minimise it with respect to controls at time t (13), satisying the constraints (10);
3. find the minimum of the quadratic form for each state and store the controls, by that the RHS of the system (5) be determined at $t - \Delta t$, where Δt is the step of Euler method;
4. then determine $\phi_{i_1,i_2,i_3}(t - \Delta t)$;
5. repeat the procedure from step 2 until $t = 0$.

Now the most cumbersome part of the procedure, that is the step 1 is realized. Give an examle of automatically generated RHS of (5) for the state $\{6,4,2\}$. Since the formula is large enough it is separated in two parts: $Q1$ and $Q2$, so that the RHS is equal to $Q1+Q2$.

$$
\begin{aligned}
Q1 = &\Big\{ -\cos(2\pi t) - \sin(2\pi t + 5/12\,\pi) - \sin(2\pi t + 1/4\,\pi) + \\
&u^{21}_{6,4,2}(t) - u^{12}_{6,4,2}(t) \Big\}^2 + \\
&\quad \Big\{ -2\,\cos(2\pi t) - 2\,\sin(2\pi t + 5/12\,\pi) - 2\,\sin(2\pi t + 1/4\,\pi) + \\
&u^{12}_{6,4,2}(t) - u^{21}_{6,4,2}(t) + u^{32}_{6,4,2}(t) - u^{23}_{6,4,2}(t) \Big\}^2 + \\
&\Big\{ -0.5\,\cos(2\pi t) - 0.5\,\sin(2\pi t + 5/12\,\pi) - 0.5\,\sin(2\pi t + 1/4\,\pi) + \\
&u^{23}_{6,4,2}(t) - u^{32}_{6,4,2}(t) \Big\}^2
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
Q2 = {}&\phi_{5,4,2}(t)\Big[-\cos(2\pi t) + 10 + u^{21}_{6,4,2}(t) \Big] + \\
&\phi_{6,3,2}(t)\Big[-2\,\cos(2\pi t) + 14 + u^{12}_{6,4,2}(t) + u^{32}_{6,4,2}(t) \Big] + \\
&\phi_{6,4,1}(t)\Big[-0.5\,\cos(2\pi t) + 6 + u^{23}_{6,4,2}(t) \Big] + \\
&\phi_{6,4,2}(t)\Big[3.5\,\cos(2\pi t) - 60 - 3.5\,\sin(2\pi t + 5/12\,\pi) - 3.5\,\sin(2\pi t + 1/4\,\pi) - \\
&\quad -2\,u^{21}_{6,4,2}(t) - 2\,u^{12}_{6,4,2}(t) - 2\,u^{32}_{6,4,2}(t) - 2\,u^{23}_{6,4,2}(t) \Big] + \\
&\phi_{6,4,3}(t)\Big[0.5\,\sin(2\pi t + 5/12\,\pi) + 6 + 0.5\,\sin(2\pi t + 1/4\,\pi) + u^{32}_{6,4,2}(t) \Big] + \\
&\phi_{6,5,2}(t)\Big[2\,\sin(2\pi t + 5/12\,\pi) + 14 + 2\,\sin(2\pi t + 1/4\,\pi) + u^{21}_{6,4,2}(t) + u^{23}_{6,4,2}(t) \Big] + \\
&\phi_{7,4,2}(t)\Big[\sin(2\pi t + 5/12\,\pi) + 10 + \sin(2\pi t + 1/4\,\pi) + u^{12}_{6,4,2}(t) \Big]
\end{aligned}
\tag{16}
$$

For each t the quadratic form $Q1+Q2$ must be minimized with respect to variables (13), satisfying the constraints (10).

3.5 Specific features of the problem

This problem has the following specific features which describe its parallelization capabilities:

- the number of equations is equal to the product of the numbers of states for each dam;
- generators of the MCs are the three-diagonal matrices, so one can use sparse matrices to economize the memory;
- due to above property the tensor products in (5) consist of small number of entries, therefore it make sense to use the sparse tensors;
- the running cost in the criterion $\mathbf{f_0}(t,u) = f_0^1(t,u) \otimes f_0^2(t,u) \otimes f_0^3(t,u)$. So it is not necessary to keep in the memory the full tensor $\mathbf{f_0}(t,u) \otimes \mathbf{X}$, and one can calculate just one entry which is necessary for particular equation, this permits to economize the computer resources as well;
- the first step of the procedure Section 3.4 might be performed in advance and does not need the repetition until changes of parameters (11), (12);
- steps 2–4 admit parallelization with shared memory for storage of $\phi_{i_1,i_2,i_3}(t)$ and $\phi_{i_1,i_2,i_3}(t - \Delta t)$. To pass from t to $t - \Delta t$ one needs to complete the calculation for all variables at t which need the synchronization of these parallel processes;
- two sets of variables $\phi_{i_1,i_2,i_3}(t)$ and $\phi_{i_1,i_2,i_3}(t-\Delta t)$ must be stored in case of using the check point mechanism. The set of controls (13) must be stored for each time $t \in [0, T]$.

4 Conclusion

The article gives the statement of the connected Markov chain control problem in tensor form. The procedure for numerical solution for the problem had been suggested, the important feature of this procedure is that it allows the parallelization. The most first cumbersome step of the procedure is realized for the system of 3 dams, the detailed analysis of such systems is the matter of future works.

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